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Freezing Transitions in Non-Fellerian Particle Systems

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Non-Fellerian particle systems are characterized by nonlocal interactions, somewhat analogous to non-Gibbsian distributions. They exhibit new phenomena that are unseen in standard interacting particle systems. We consider freezing transitions in one-dimensional non-Fellerian processes which are built from the abelian sandpile additions to which in one case, spin flips are added, and in another case, so called anti-sandpile subtractions. In the first case and as a function of the sandpile addition rate, there is a sharp transition from a non-trivial invariant measure to the trivial invariant measure of the sandpile process. For the combination sandpile plus anti-sandpile, there is a sharp transition from one frozen state to the other anti-state.

KEY WORDS: Sandpile dynamics, interacting particle systems, non-Fellerian processes, nonequilibrium phase transitions

1. INTRODUCTION

Interacting particle systems (IPS) are Markov processes with a spatial degree of freedom. They consist of a large number of discrete components that are each stochastically updated depending on the state of the others. Standard references include Refs. 4, 7, 13 and 15. IPS appear in a large variety of applications but their basic features have been shaped somewhere at the boundary between statistical mechanics and probability theory. In statistical physics we know them mostly as Glauber and Kawasaki processes as in the stochastic or kinetic Ising model, but also such models as the contact and the voter model have found a wide audience.

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Dynamical studies of phase transitions, of metastability, of hydrodynamic scaling etc. have been undertaken in the framework of IPS. Much of the motivation in the study of IPS has however always come from the search for new phenomena. The so called sandpile processes have in that sense been seen as an important addition.[2] In the present paper, we investigate phase transitions that are directly related to an important feature of such models, their nonlocality. That nonlocality means that the updating of the configuration in some specific area can depend (without much decay) on the previous configuration very far away from that area. Mathematically, that property is referred to as the nonFeller property, where the transfer operator does not leave invariant the set of continuous functions. For a recent review concerning mathematical results on the thermodynamic limit of sandpile models in spite of that nonlocality, see Refs. 10 and 14.

We will not stay with one specific model but we will use the sandpile model as a starter to build various scenarios of freezing transitions. The abelian sandpile model has been widely studied in the context of so-called self-organized criticality. Its nonlocality, which is itself directly related to an infinite scale separation between a reaction and a diffusion mechanism, is indeed probably very important for power-law statistics of the avalanche size. From the point of view of probability theory, it is the best known case of a spatially extended non-Fellerian stochastic dynamics.[8] It has challenged our basic understanding of the construction of interacting processes in infinite volume, even in one dimension.

In one dimension, the stationary measure of the standard abelian sandpile model is trivial in the thermodynamic limit. However the dynamics of relaxation to this measure is non-trivial. In Ref. 9 we have constructed the dynamics for the one-dimensional sandpile model on the infinite lattice $\mathbb{Z}$. The result is a monotone non-Fellerian process which converges in finite time to its unique stationary state, which is concentrating on the maximal configuration. As soon as one changes to other lattices, such as decorated one-dimensional lattices, the triviality of the limiting stationary measure disappears, and especially in one dimension existence of thermodynamic limits is not guaranteed due to the presence of infinite avalanches.[5]

In the present paper, we combine the one-dimensional sandpile model with a spin-flip dynamics (pure spin flip as well as Glauber type or more general spin flip processes with positive rates). Indeed, in one dimension, the standard sandpile model has only two possible heights per site, and spin flip just means changing the height from one to the other. In the language of sandpiles, adding a pure spin flip is the simplest example of combining two different toppling mechanisms, the spin-flip part corresponding to a purely dissipative (diagonal) toppling matrix. More precisely, in Sec. 3, our dynamics has a formal generator

$$\begin{aligned} Lf(\eta) &= \alpha L_{\text{sandpile}} f(\eta) + L_{\text{flip}} f(\eta) \\ &= \alpha \sum_{x\in\mathbb{Z}} [f(a_x\eta) - f(\eta)] + \sum_{x\in\mathbb{Z}} c(x,\eta)\,[f(\theta_x\eta) - f(\eta)] \end{aligned} \tag{1.1}$$

where a_x denote the abelian sandpile addition operators, and θ_x the flip operator, on configurations $\eta \in \{1, 2\}^{\mathbb{Z}}$. In words that means that at rate α we add and stabilize according to the abelian sandpile rule, and at rate $c(x, \eta)$, we just flip the value of the height where we added. In that way we have a parameter α that describes the relative weight of the sand additions versus the spin flips. The resulting "sand-flip" dynamics shows a freezing transition as a function of that α. Observe that the rates remain bounded away from zero. In the simplest case where $c(x, \eta) = 1$ (adding pure spin flip), our main result says that for $\alpha \geq \alpha_c = 1$ there is a finite time after which the system reaches the maximal configuration (i.e., the sandpile part "wins"), whereas the unique stationary measure is non-trivial and mixing under spatial translations for $\alpha < 1$. That is a strong manifestation of the nonlocality of the dynamics. Indeed, for Fellerian processes such phenomenon cannot occur: the freezing state is invariant for the sandpile part of the generator but not for the spin-flip part. We prove by almost explicit construction that this freezing transition occurs for general bounded and positive spin-flip rates $c(x, \eta)$, and we give an explicit example where the effect of a non-trivial interaction in the rates on the value of the transition point α_c can be computed.

As a second example of such "competition between generators," in Section 4 we consider a combination of a sandpile and an anti-sandpile process. This dynamics is inspired by Ref. 6. The anti-sandpile part of the dynamics consists of removing grains ("adding holes") and stabilizing by reverse topplings. The infinite volume limit of the anti-sandpile stationary measure is a Dirac measure concentrating on the minimal configuration. Our main result here is that unless the rates of addition and subtraction are equal, the limiting stationary measure is a Dirac measure corresponding to the dominant rate. We thus have a sharp transition between two different frozen states.

These nonequilibrium phase transitions have an interest of their own but they also go some way in adding extra and physically relevant interactions to the standard abelian sandpile. We have in mind the sticky sandpiles of Ref. 3, for which our dynamics is a subclass, and for which various transitions have been numerically checked. Our freezing transition was also and for the first time seen in a type of queueing model,[(12)] which was built in strong analogy with the constructions of the one-dimensional sandpile process in Ref. 9. As a final note, it is interesting to make the analogy with non-Gibbsian measures. In some sense, they are the "equilibrium analogue" of non-Fellerian interacting particle systems. In Ref. 11 an example of a freezing transition was obtained, strongly connected again with the absence of continuity of the local conditional probabilities.

2. NON-FELLERIAN PROCESSES

We start with a short introduction to the somewhat unusual mathematical features of non-Fellerian processes. They are important to keep in mind for what follows.

We take the state space of our process to be $\Omega = \{1, 2\}^{\mathbb{Z}}$. One can think of having either height one or height two at each site of the integer lattice. Two configurations are close by (have a small distance) if they coincide on a large interval. Local functions $f(\eta)$ only depend on a finite number of coordinates in $\eta \in \Omega$. Continuous functions are those that can be approximated by local functions.

In the theory of IPS, an important tool is the semigroup $S(t)$ of the process. For every time $t \geq 0$, it gives the transfer operator $f \mapsto S(t)f$ on the level of functions (or, observables) f. A Feller-semigroup maps continuous functions into continuous functions. If we have a local function f, then $S(t)f$ would still be continuous. For the process, it means that for example the height at the origin depends on the previous (in time) configuration mostly in a finite region. That gets also reflected in the relation with the generator L of the process. If $S(t)$ is a Feller-semigroup, then there exists a uniformly dense set of continuous functions which are in the domain of the generator, i.e., for which

$$\lim_{t \downarrow 0} \|(S(t)f - f)/t - Lf\|_\infty = 0. \tag{2.1}$$

This implies the right-continuity of the semigroup. Non-Fellerian processes need not satisfy that.

As another example, usually in standard IPS one "finds" a stationary distribution μ by solving $\int Lf d\mu = 0$ (arbitrary f) for μ. For non-Fellerian processes, a stationary distribution μ need not satisfy $\int Lf d\mu = 0$ for all local functions f, even though $\mu S(t) = \mu$. Another way to "find" a stationary distribution in IPS is to look at the (time-) asymptotic behavior. One then follows the time-evolution of the distribution which, one hopes, would settle in a stationary distribution after some transient regime. To see what can happen for a non-Fellerian process, consider the following example. Starting from a configuration $\eta \in \Omega$, we flip a 1 to 2 at rate 1, independently for all lattice sites, and if the configuration is $\overline{2}$ (all two), then we flip it at rate one to the minimal configuration $\eta \equiv 1$, denoted by $\overline{1}$. That is a non-Fellerian process with convergence $\mu S(t) \to \delta_2$ (where δ_2 is the Dirac measure concentrating on the configuration $\overline{2}$) for all translation invariant measures $\mu \neq \delta_2$, but clearly, δ_2 is not invariant. In fact, that process has no invariant measure!

3. ADDING SPIN FLIPS TO THE SANDPILE PROCESS

The state space of our process is $\Omega = \{1, 2\}^{\mathbb{Z}}$. For a configuration $\eta \in \Omega$, $\eta(x) \in \{1, 2\}$ is usually interpreted as the height or the number of grains at site x. That language can be continued even when combining the sandpile automaton with other dynamics but we prefer to use the words "active" for $\eta(x) = 2$ and "inactive" for $\eta(x) = 1$.

The dynamics will change the configuration according to a combination of the standard sandpile model and a spin flip dynamics. We start with the simplest form of spin flip, changing "active" into "inactive" and vice versa at rate 1: the spin flip θ_x is thus defined as

$$(\theta_x\eta)(y) = \begin{cases} 3-\eta(x), & \text{if} \quad y = x \\ \eta(y), & \text{if} \quad y \neq x \end{cases} \tag{3.1}$$

Only in Sec. 3.4 will we generalize the spin flip part of the dynamics.

For the sandpile dynamics, we can rely on our previous work in Ref. 9 where we have studied the infinite volume limit of the one-dimensional sandpile process. We will therefore not bother to redo the limiting procedures but below we immediately give the result, the form of the infinite volume addition operators a_x. The informal verbal prescription of the sandpile dynamics goes as follows: if a site is inactive, it becomes active at rate α. If the site x is already active, one looks left and right of x at the closest sites x_η^- and x_η^+ which are inactive. Again at rate α these two become active and the mirror image of x with respect to the middle of $[x_\eta^-, x_\eta^+]$ becomes inactive. That corresponds, in the infinite volume limit, to the result (in finite volume) of adding and stabilizing through a sequence of topplings, where upon a single toppling of a site the site looses two grains and gives one grain to each neighbor, except if the site is at the boundary where there is only one neighbor receiving a grain. See Ref. 9 and 14 for more details on the abelian sandpile model in $d = 1$.

The infinite volume addition operator a_x is defined more precisely as follows. For $\eta \in \Omega$ and $x \in \mathbb{Z}$ with $\eta(x) = 1$, we have $a_x\eta = \eta + e_x$ (where $e_x(x) = 1$ and $e_x(y) = 0$ otherwise), i.e., inactive becomes active, or, the height one at x simply changes to height two (and no other changes). For $\eta \in \Omega$ and $x \in \mathbb{Z}$ with $\eta(x) = 2$ we look at the right – respectively at the left – of x to find the first site x_η^+ (if that site does not exist we put $x_\eta^+ = \infty$), – respectively x_η^- (if that site does not exist we put $x_\eta^- = -\infty$) – with $\eta(x_\eta^+) = \eta(x_\eta^-) = 1$. We then define

$$(a_x\eta)(y) = \begin{cases} 1 \quad \text{if} \quad y = x_\eta := x_\eta^+ + x_\eta^- - x \\ 2 \quad \text{if} \quad y \neq x_\eta, \quad \text{and} \quad x_\eta^- \leq y \leq x_\eta^+ \\ \eta(y) \text{ otherwise} \end{cases} \tag{3.2}$$

if both x_η^+ and x_η^- exist. In words, upon adding one unit at x, the first sites at height one to the left (x_η^-) and to the right of x (x_η^+) become sites with height 2, and the site which is the mirror image of x with respect to the middle of x_η^+ and x_η^- becomes of height one, all other sites remain unaltered. We have to extend that definition to cases where one of the sites x_η^+, x_η^- does not exist (i.e., when there is no site to the right or to the left of x having height one). That is done by taking the limit with "boundary condition 1," i.e., if at least one of the $x_\eta^\pm$ is infinite, then

$$(a_x\eta)(y) = \begin{cases} 2 & \text{if} \quad x_\eta^- \leq y \leq x_\eta^+ \\ \eta(y) & \text{otherwise} \end{cases} \tag{3.3}$$

Remark that (3.1) is a special case of "addition" with "purely dissipative toppling," i.e., upon toppling an active site two grains disappear (diagonal toppling matrix). In that sense combination of a_x and θ_x is the simplest example of combining two different toppling mechanisms (matrices) in one process.

3.1. Construction

Intuitively, our process is governed by two independent collections of Poisson processes, $N_t^{x,f}$, $N_t^{x,a}$, indexed by sites $x \in \mathbb{Z}$, and independent for different sites. On the event times of $N_t^{x,a}$ we apply the addition operator, and on the event times of $N_t^{x,f}$ we "flip" the state, i.e., we apply θ_x. We put the rate of the "sandpile-clocks" equal to α, and the rate of the "flip-clocks" equal to one. Formally, our process has as a generator on local functions $f : \Omega \to \mathbb{R}$,

$$\begin{aligned} Lf(\eta) &= \alpha \sum_{x\in\mathbb{Z}} (f(a_x\eta) - f(\eta)) + \sum_{x\in\mathbb{Z}} (f(\theta_x\eta) - f(\eta)) \\ &:= \alpha L_S f(\eta) + L_F(\eta) \end{aligned} \tag{3.4}$$

where L_S stands for "sandpile generator" and L_F for "flip-generator."

To show that there exists a Markov process with càdlàg-paths corresponding to the Poisson process description above or to the formal generator (3.4), we use a monotonicity argument analogous to the one in Ref. 9. We repeat the main steps and the minor modifications to be done here. First we define the action of a_x on η as a "birth" if $\eta(x) = 1$, and as an avalanche if $\eta(x) = 2$, whereas the action of θ_x is (of course) also called a birth if $\eta(x) = 1$ (leading to $\theta_x(\eta(x)) = 2$), and a "death" if $\eta(x) = 2$ (leading to $\theta_x(\eta(x)) = 1$). We can then split the formal generator in three parts:

$$L = L_a + L_b + L_d \tag{3.5}$$

where

$$\begin{aligned} L_a f(\eta) &= \alpha \sum_{x\in\mathbb{Z}} \chi(\eta(x) = 2)(f(a_x\eta) - f(\eta)) \\ L_b f(\eta) &= (1+\alpha) \sum_{x\in\mathbb{Z}} \chi(\eta(x) = 1)(f(\theta_x\eta) - f(\eta)) \\ L_d f(\eta) &= \sum_{x\in\mathbb{Z}} \chi(\eta(x) = 2)(f(\theta_x\eta) - f(\eta)) \end{aligned} \tag{3.6}$$

Here, $\chi(.)$ denotes the indicator function. The construction is then as follows:

- Construct a process corresponding to $L_a + L_d$ (only avalanches and deaths) on the set Ω_f of configurations with a finite number of sites with height 2. That is a (non-explosive) countable state space Markov chain on the set of finite subsets of $\mathbb{Z}$. Show by coupling that that process is *monotone*. The coupling is identical to that of Ref. 9 for the avalanche events. For the deaths: we let two twos die together if possible, and otherwise independently.
- Construct a process corresponding to $L_a + L_d$ with births in a finite interval, i.e., having generator

$$L_n f(\eta) = (L_a + L_d) f(\eta) + (1+\alpha) \sum_{x=-n}^{n} \chi(\eta(x) = 1)(f(\theta_x \eta) - f(\eta))$$

  We construct that process once more as a countable state space Markov chain, and show that it is monotone. Its semigroup e^{tL_n} is denoted by $S_n(t)$. Moreover, we have the following monotonicity as a function of the interval on which we allow births: for all $t > 0$, $n \in \mathbb{N}$, f monotone, $\eta \in \Omega_f$,

$$(S_n(t)f)(\eta) \leq (S_{n+1}(t)f)(\eta)$$

- For general monotone f and $\eta \in \Omega$ arbitrary:

$$S(t)f(\eta) := \sup_{n \in \mathbb{N}} \sup_{\eta \in \Omega_f} S_n(t) f(\eta) \tag{3.7}$$

The process obtained by the above construction is called the SF-process (sand-flip process). We denote its path space measure starting from η by $\mathbb{P}_\eta$.

3.2. Basic Properties

Besides monotonicity, the SF-process has very similar "quasi-Feller" properties as the one-dimensional sandpile process of Ref. 9. In particular, we have the following analogue of Theorem 5.1 of Ref. 9. Let us denote by $\Omega' \subset \Omega$ the configurations with an infinite number of ones to the left and to the right of the origin. We then enumerate $\eta^{-1}\{1\} = \{X_i(\eta), i \in \mathbb{Z}\}$ where $X_0(\eta) := \min\{x \geq 0 : \eta(x) = 1\}$, and the other X_i are in increasing order the sites where $\eta(x) = 1$. The X_i define the η-dependent disjoint intervals $I_i = (X_{i-1}(\eta), X_i(\eta)]$. A function is called N-local if it depends on the heights $\eta(i)$ for $i \in \cup_{j=-N}^{N} I_j$. Every local function is N-local, but a N-local function can be non-local, e.g. $f(\eta) = e^{-|X_1(\eta)|}$ is bounded 1-local, but non-local. The idea is that the natural space to define the action of iterates of the generator is the set of N-local functions. That is made precise in the following definition and proposition.

Definition 1. A configuration $\eta \in \Omega'$ is called decent if

$$a(\eta) = \limsup_{n\to\infty} \frac{1}{2n} \sum_{i=-n}^{n} |X_i(\eta) - X_{i-1}(\eta)| < \infty \tag{3.8}$$

The set of decent configurations is denoted by Ω_{dec}.

Proposition 1. *Let $\eta \in \Omega_{\text{dec}}$, f be bounded and N-local, then for $t < 1/[4(1+\alpha)ea(\eta)]$, the series $\sum_{n=0}^{\infty}[t^n(L^n f)(\eta)]/(n!)$ converges absolutely and equals $S(t)f(\eta)$, where $S(t)$ is the semigroup of the process defined above. In particular*

$$\lim_{t\to 0} \frac{S(t)f(\eta) - f(\eta)}{t} = Lf(\eta) \tag{3.9}$$

i.e., L is the "pointwise generator" of the process.

Proof: The same proof of Ref. 9 can be used, if one notices that the extra "death" part of the generator can only split one of the intervals I_i into smaller ones, by creating an extra 1. This implies that if f is N-local, then $[f(a_i\eta) - f(\eta)] = 0$ for all $i \in \mathbb{Z} \setminus \cup_{j=-N-1}^{N+1} I_j$. Therefore Lf depends only on the heights in $\cup_{j=-N-1}^{N+1} I_j$. Iterating the argument, one sees that $L^n f$ depends only on height in $\cup_{j=-N-n}^{N+n} I_j$, and one recovers the same estimate

$$\|(L^n)f\|_\infty \leq \prod_{k=0}^{n} \left(\sum_{i=0}^{N+k} |I_i| \right) 2^n (1+\alpha)^n \|f\|_\infty \tag{3.10}$$

which gives the result of the proposition, by application of Lemma 4.1 in Ref. 9. □

The following result, analogous to Corollary 6.1 in Ref. 9 and to Proposition 3.1 in Ref. 12, shows that the process is always non-Feller.

Proposition 2. *For all $\alpha > 0$, the SF-process is non-Feller.*

Proof: We denote by $\overline{2}$ the maximal configuration $\eta \equiv 2$. We define the configuration η_{spec} by

$$\eta_{\text{spec}}(x) = \begin{cases} 1 \text{ if } x = 0 \\ 2 \text{ otherwise} \end{cases} \tag{3.11}$$

Then one shows as in Ref. 12 that for $f_0(\eta) = \eta(0)$

$$\lim_{t\to 0, t>0} S(t) f_0(\eta_{\text{spec}}) = 2 \tag{3.12}$$

i.e., by the avalanche part of the dynamics, the isolated 1 is turned "immediately" into a 2. Therefore, the right limit of η_t as $t \to 0, t > 0$ is almost surely equal to $\overline{2}$ when we start from η_{spec}.

This lack of right-continuity contradicts the Feller property. A dense set of continuous functions such as in (2.1) contains a function f such that

$$f(\eta_{\text{spec}}) \neq f(\overline{2}) \tag{3.13}$$

Combination of (3.12), (2.1), and (3.13) gives a contradiction. □

3.3. Stationary Measure

Denote by $\mathcal{I}$ the set of invariant probability measures of the SF-process defined in the previous section. Let $\mathcal{S}$ be the set of translation invariant probability measures on Ω. We denote by δ_1, resp. δ_2 the Dirac measure concentrating on the configuration $\eta \equiv 1$ resp. $\eta \equiv 2$.

By monotonicity of the process $\mathcal{I}$ contains the limits

$$\nu_i := \lim_{t\to\infty} \delta_i S(t) \tag{3.14}$$

so, in particular, $\mathcal{I}$ is not empty. In fact we have

Theorem 1. *For all $\alpha > 0$, $\mathcal{I} = \{\mu_\alpha\}$ and for all probability measures ν on Ω,*

$$\lim_{t\to\infty} \nu S(t) = \mu_\alpha \tag{3.15}$$

Moreover,

(a) for $\alpha < 1$, the density of sites with height one is given by

$$\int \chi(\eta(0) = 1) d\mu_\alpha = \frac{1-\alpha}{2} \tag{3.16}$$

and μ_α is a translation invariant measure which is mixing under translations and non-product.

(b) For $\alpha \geq 1$,

$$\mu_\alpha = \delta_2 \tag{3.17}$$

Moreover for $t > [\log(\alpha+1) - \log(\alpha-1)]/2$ and for every $\eta \in \Omega$,

$$\mathbb{P}_\eta(\eta_t(0) = 2) = 1.$$

Proof: We start with the following lemma.

Lemma 1. *Let μ be a probability measure on Ω that is mixing under spatial translations, with $\int \chi(\eta(0) = 1) d\mu = \rho > 0$. Then we have*

(a) If $t < \rho/[4(1+\alpha)e]$, then $\mu S(t)$ is mixing under spatial translations.
(b) Let $t(\alpha, \rho)$ be the solution of

$$\rho e^{-2t} + \frac{1-\alpha}{2}(1 - e^{-2t}) = 0 \tag{3.18}$$

when it exists, otherwise by definition $t(\alpha, \rho) = +\infty$. Then we have for all $t < t(\alpha, \rho)$, $\mu S(t)$ is mixing under spatial translations and

$$\frac{d}{dt}\int \eta(0) d(\mu S(t)) = \int L\eta(0) d(\mu S(t)) \tag{3.19}$$

$$\rho(t) := \int \chi(\eta(0) = 1) d(\mu S(t)) = \rho e^{-2t} + \frac{1-\alpha}{2}(1 - e^{-2t}) \tag{3.20}$$

Proof: (a) Is exactly as in Ref. 9, Lemma 6.1.
(b) Let $t_0 = \inf\{\rho/[5(1+\alpha)e], t(\alpha, \rho)\}$; by a), (3.19) holds for $t \leq t_0$. Denote

$$k^+(i, \eta) = \inf\{j \geq 0 : \eta(i + j) = 1\} \tag{3.21}$$

$$k^-(i, \eta) = \inf\{j > 0 : \eta(i - j) = 1\} \tag{3.22}$$

We compute

$$L\eta(0) = \alpha\chi(\eta(0) = 1)(k^+(1, \eta) + k^-(0, \eta) + 1) + 3 - \alpha - 2\eta(0) \tag{3.23}$$

For $\nu \in \mathcal{S}$ (see Ref. 9, (6.77)),

$$\int \chi(\eta(0) = 1) k^-(0, \eta)\, d\nu = \int \chi(\eta(0) = 1)(k^+(1, \eta) + 1)\, d\nu = 1$$

so that

$$\begin{aligned}\int L\eta(0) d(\mu S(t)) &= \alpha + 3 - 2\int \eta(0)\, d(\mu S(t)) \\ &= \alpha - 1 + 2\int \chi(\eta(0) = 1)\, d\mu S(t)\end{aligned} \tag{3.24}$$

and hence

$$\frac{d\rho(t)}{dt} = -\alpha + 1 - 2\rho(t)$$

which gives (3.20) for $t < t_0$. If $t_0 \neq t(\alpha, \rho)$, then we can start the reasoning anew and iterate from t_0 to $t_1 = \min\{\rho(t_0)/[5(1+\alpha)e], t(\alpha, \rho(t_0))\}$, with new initial distribution $\mu S(t_0)$. As a consequence, for $0 \leq t \leq t_0 + t_1$, $\rho(t)$ is still given by (3.20), etc. □

By monotonicity, the invariant measures defined in (3.14) satisfy $\nu_1 \leq \nu_2$. Moreover, the process is totally ergodic (in the sense of Ref. 7, chapter 1, Definition 1.9), i.e., $\mathcal{I}$ is a singleton if and only if $\nu_1 = \nu_2$. In that case $\nu_1 = \nu_2$ is also mixing under spatial translations, see Ref. 1 Theorem 1.4 (ii); indeed, the proof of the latter is a computation that does not involve the Feller property of the considered Markov semi-group, and that relies on the following property (proven as in Ref. 9, Lemma 6.1).

$$\lim_{|x|\to\infty} \int |S(t)[f\tau_x g] - S(t)fS(t)(\tau_x g)|d\mu = 0$$

for all local functions f, g on Ω, where τ_x denotes the spatial shift by $x \in \mathbb{Z}$.

First consider $\alpha < 1$. Then $t(\alpha, \rho) = \infty$ for all $\rho > 0$. Let λ_ρ denote the translation invariant product measure on Ω with $\lambda_\rho(\eta(0) = 1) = \rho$. Then $\lambda_\rho \uparrow \delta_2$ when $\rho \downarrow 0$. Therefore, using monotonicity and item b) of Lemma 1, we obtain for all $t \geq 0$

$$\begin{aligned}\int \chi(\eta(0) = 1)\, d(\delta_2 S(t)) &= \lim_{\rho\downarrow 0} \int \chi(\eta(0) = 1)\, d(\lambda_\rho S(t)) \\ &= \lim_{\rho\downarrow 0} \left(\rho e^{-2t} + \frac{1-\alpha}{2}(1 - e^{-2t}) \right) \\ &= \frac{1-\alpha}{2}(1 - e^{-2t}) \end{aligned} \tag{3.25}$$

Therefore, in the notation (3.14),

$$\nu_2(\eta(0) = 1) = \frac{1-\alpha}{2} \tag{3.26}$$

On the other hand starting from δ_1, we have $t(\alpha, \rho) = t(\alpha, 1) = \infty$, and we obtain from item b) of Lemma 1

$$\lim_{t\to\infty} \int \chi(\eta(0) = 1)\, d(\delta_1 S(t)) = \lim_{t\to\infty} \left(e^{-2t} + \frac{1-\alpha}{2}(1 - e^{-2t}) \right) = \frac{1-\alpha}{2} \tag{3.27}$$

$$\nu_1(\eta(0) = 1) = \nu_2(\eta(0) = 1) = \frac{1-\alpha}{2} \tag{3.28}$$

and combining that with $\nu_1 \leq \nu_2$, gives $\nu_1 = \nu_2$.

To see that μ_α is not a product measure, observe that because $\mu_\alpha \in \mathcal{S}$, it can be product only if $\mu_\alpha = \lambda_\rho$ with $\rho = (1-\alpha)/2$. Therefore, for all f local $\int Lf d\lambda_\rho = 0$. Indeed λ_ρ concentrates in decent configurations, so we are allowed to use the generator by Proposition 1. Notice that in the sandpile part of the dynamics two or more neighboring ones can never be created. Therefore, one computes the action of the sandpile part of the generator on the function

$H_n(\eta) = \chi(\eta(1) = \cdots = \eta(n) = 1)$ with $n \geq 2$, which gives after integration over the product measure

$$\int L_S H_n d\lambda_\rho = -n\rho^n - 2\sum_{i=0}^{\infty} i(1-\rho)^i \rho^{n+1} = -n\rho^n - 2\rho^{n-1}(1-\rho) \quad (3.29)$$

For the spin-flip part one has

$$\int L_F(H_n) d\lambda_\rho = -n\rho^n + n\rho^{n-1}(1-\rho) \quad (3.30)$$

Therefore, for the combined dynamics, the condition that λ_ρ is stationary leads to

$$\int (\alpha L_S + L_F)(H_n)\, d\lambda_\rho = 0 = \rho^{n-1}\,(-n\rho + n(1-\rho) - n\alpha\rho - 2\alpha(1-\rho)) \quad (3.31)$$

which gives (for $n \geq 2$)

$$\rho = \frac{n - 2\alpha}{2n + (n-2)\alpha} \quad (3.32)$$

Notice that Eq. (3.31) is *not* valid for $n = 1$, because a *single one can* be created in the sandpile dynamics, so an extra term $(1 - \rho)$ should be added for $n = 1$. Since 3.32 should be valid for all $n \geq 2$, we obtain a contradiction. Hence, the invariant measure is indeed non-product.

For $\alpha = 1$, (3.25) is still valid (indeed, $t(1, \rho) = \infty$ for all $\rho \in (0, 1)$) and this gives

$$\int \chi(\eta(0) = 1)\, d(\delta_2 S(t)) = 0$$

Therefore $\delta_2 S(t) = \delta_2$. On the other hand, item b) of Lemma 1 gives

$$\lim_{t\to\infty} \int \chi(\eta(0) = 1)\, d(\delta_1 S(t)) = 0$$

Therefore $\nu_1 = \delta_2$, hence we obtain $\nu_1 = \nu_2 = \delta_2$.

Finally, consider $\alpha > 1$. Then we have

$$t(\alpha, 1) = \frac{1}{2}\log\left(\frac{\alpha+1}{\alpha-1}\right) \quad (3.33)$$

Hence, for all $\varepsilon > 0$ there exists $t_0 < t(\alpha, 1)$ such that for all $t > t_0$

$$\int \chi(\eta(0) = 1)\, d(\delta_1 S(t)) < \varepsilon$$

Therefore, $\nu_1 = \lim_{t\to\infty} \delta_1 S(t) = \delta_2$, and we conclude from the inequalities $\nu_1 = \delta_2 \leq \nu_2$, and $\nu_2 \leq \delta_2$ that $\nu_1 = \nu_2 = \delta_2$.

Moreover, we obtain that from any initial distribution ν, the limiting measure δ_2 is reached in finite time $T_\nu \leq t(\alpha, 1)$. □

Remark 1. *For $\alpha \geq 1$, we have $\int Lgd\mu_\alpha \neq 0$ for non constant g, since for the sandpile part $\int L_S gd\mu_\alpha = 0$, whereas for the flip part $\int L_F gd\mu_\alpha \neq 0$. Therefore, in that case the invariant measure cannot be found by solving $\int Lfd\mu = 0$ for μ, which gives another argument for the non-Fellerian character of the SF-process, see also under Sec. 2.*

3.4. Robustness of the Freezing Transition

In this section we consider more general local perturbations of the sandpile generator. We show that the freezing phenomenon, i.e., having δ_2 as unique invariant measure for α large enough, and a non-trivial invariant measure for α small persists.

More precisely, we consider a formal generator of the type

$$L = \alpha L_S + L_G \tag{3.34}$$

where L_G is the generator of a spinflip dynamics (i.e., with possibly configuration dependent rates):

$$L_G f(\eta) = \sum_{x \in \mathbb{Z}} c(x, \eta)(f(\theta_x \eta) - f(\eta)) \tag{3.35}$$

where the flip-rates $c(x, \eta)$ are supposed to be translation invariant, local and bounded from below. Therefore,

$$m \leq c(x, \eta) \leq M \tag{3.36}$$

for some $0 < m \leq M < \infty$ independent of η.

To define this process, we use a series expansion as in Proposition 1 to *define* the semigroup. Remark that since we do not assume that L_G is the generator of a monotone process, the semigroup cannot be constructed by monotonicity. Instead, $S(t)f(\eta)$ is defined by the series expansion as long as the configuration η is decent, and contrary to the monotone case, this cannot necessarily be extended to non-decent configurations such as the maximal configuration $\overline{2}$.

We then have the following

Theorem 2. *Consider the process with formal generator (3.34). We have*

1. *For $\alpha < m$, δ_2 is not an invariant measure. In fact, if $\mu \in \mathcal{S}$ is an invariant measure, then*

$$\mu(\eta(0) = 1) \geq \frac{m - \alpha}{2M} \tag{3.37}$$

2. *If $\alpha > M$, then for all $\mu \in \mathcal{S}$, with $\mu(\eta(0) = 1) > 0$, $\mu S(t) \to \delta_2$ as $t \to \infty$. Therefore, δ_2 is the only possible invariant measure.*

Proof: Let $\mu \in \mathcal{S}$ be such that $\mu(\eta(0) = 1) > 0$. We denote $\rho_t = (\mu S(t))(\eta(0) = 1)$. Then by the obvious generalization of Lemma 1, we write

$$\frac{d\rho_t}{dt} = -\alpha - 2\int \chi(\eta(0) = 1)c(0, \eta)\, d(\mu S(t)) + \int c(0, \eta)\, d(\mu S(t)) \tag{3.38}$$

Therefore

$$-\alpha - 2M\rho_t + m \leq \frac{d\rho_t}{dt} \leq -\alpha - 2m\rho_t + M \tag{3.39}$$

Hence, if $\alpha < m$ and $\rho_t < (m - \alpha)/2M$,

$$\frac{d\rho_t}{dt} > 0$$

so there can be no invariant measure μ with $\rho = \mu(\eta(0) = 1) < (m - \alpha)/2M$. That proves the first item of the theorem. For the second item, if $\alpha > M$ and if $\rho_t > 0$, (3.39) gives

$$\frac{d\rho_t}{dt} < 0$$

and hence there cannot be an invariant measure with $\rho = \mu(\eta(0) = 1) > 0$. □

Remark 2. *Even if $\mu S(t)$ converges to δ_2 for all $\mu \in \mathcal{S}$ with $\mu(\eta(0) = 1) > 0$, we cannot conclude that δ_2 is an invariant measure, because the process is not Feller, see under Sec. 2.*

If L_G is the generator of a monotone process, then more precise results can be obtained. For that case we will stick to an explicit example where once more an explicit closed equation for the density can be obtained. More precisely, we consider the flip rates

$$c(x, \eta) = 1 - \gamma f_x(\eta)(f_{x-1}(\eta) + f_{x+1}(\eta)) \tag{3.40}$$

where

$$f_x(\eta) = 1 - 2\chi(\eta(x) = 1)$$

These rates correspond to the standard Glauber choice for

$$\gamma = \frac{1}{2}\tanh(2\beta) \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

where β denotes the inverse temperature (here without any meaning except for an effective coupling constant). We then have the following analogue of Theorem 1.

Theorem 3. *For the process with formal generator (3.34), and rates (3.40) we have*

1. *For* $\alpha < \alpha_c = 1 - 2\gamma$, *there exists a unique non-trivial invariant measure* μ_α *with*

$$\mu_\alpha(\eta(0) = 1) = \frac{1}{2}\left(1 - \frac{\alpha}{\alpha_c}\right) \tag{3.41}$$

2. *For* $\alpha \geq \alpha_c$, δ_2 *is the unique invariant measure.*

Proof: Since the rates (3.40) satisfy Definition 2.1 of chapter 3 in Ref. 7, the process with generator L_G is monotone. Therefore, we can construct the process with generator (3.34) by monotonicity as in Section 3.1, where we replace the coupling for the birth and death part by basic coupling. In particular the thus obtained generalized SF-process is monotone. For $\mu \in \mathcal{S}$ with $\mu(\eta(0) = 1) > 0$ we obtain

$$\frac{d\rho_t}{dt} = -\alpha + (1 - 2\rho_t)(1 - 2\gamma) \tag{3.42}$$

and from that equation, combined with monotonicity we can proceed as in the proof of Theorem 1. □

Remark 3. *As one would expect intuitively, the critical value* α_c *is decreasing in* γ, *i.e., the freezing is enhanced by stronger coupling.*

Another simple choice in which we explicitly see the effect on α_c *is obtained by adding a bias to the spin flip. Then, (3.40) becomes*

$$c(x, \eta) = 1 - \kappa f_x(\eta) = (1 - \kappa)\chi(\eta(x) = 2) + (1 + \kappa)\chi(\eta(x) = 1)$$

and a similar calculation yields the same result but with a critical value that now equals $\alpha_c = 1 - \kappa$.

4. ADDING "ANTI-ADDITIONS" TO THE SANDPILE PROCESS

4.1. The Anti-Sandpile Model

In words, the anti-sandpile process is a process where grains are removed from a configuration $\eta \in \Omega$, and afterwards (if necessary) the configuration is stabilized instantaneously by *reversed topplings*.

We first define the finite-volume process, in $[-N, N]$. If after removing grains the height is zero at one or more sites $x \in [-N, N]$, then the configuration stabilizes by a sequence of reversed topplings. Upon a reversed toppling of a site $x \in [-N, N]$ the site *gains* two grains and each of its neighbors (in $[-N, N]$) looses one grain. This means that in a reversed toppling, the boundary sites act as *a source* (instead of a sink in the ordinary toppling rule). The anti-addition operator $a_x^\dagger$ is then defined as the stable result of the subtraction of one unit at site x and performing reversed topplings until the configuration is *stable* (i.e., height everywhere 1 or 2) again.

Remark 4. *The anti-addition operator should not be confused with the inverse a_x^{-1} of the addition operator a_x. In fact, if η is recurrent,* and *$a_x^\dagger \eta$ is recurrent, then $a_x^\dagger \eta = a_x^{-1}(\eta)$, but $a_x^\dagger \eta$ need not be recurrent even if η is.*

In finite volume, the generator of the anti-sandpile process is given by

$$L^\dagger = \sum_{x=-N}^{N} (a_x^\dagger - I) \tag{4.1}$$

where I denotes the identity operator. Remark that $a_x^\dagger = \theta a_x \theta$ and

$$L^\dagger = \Theta L \Theta \tag{4.2}$$

where Θ is "global spinflip" on functions and θ is "global spinflip" on configurations, i.e.,

$$(\Theta f)(\eta) = f(\theta\eta) \tag{4.3}$$

with $\theta\eta(x) = 3 - \eta(x)$. Therefore the extension of the process generated by $L^\dagger$ to infinite volume is immediate. Its semigroup is given by

$$S(t)^\dagger = \Theta S(t)\Theta \tag{4.4}$$

where $S(t)$ is the semigroup of the sandpile process.

In infinite volume, the "anti-addition operator" $a_x^\dagger$ is then defined via

$$a_x^\dagger \eta = (\theta a_x \theta)(\eta)$$

Similarly to (3.21)–(3.22), we introduce

$$l^\pm(x, \eta) = k^\pm(x, \theta\eta)$$

and the intervals $J_i(\eta) = I_i(\theta\eta)$.

4.2. The SA Process

We now define the SA-process (i.e., "sandpile + anti-sandpile") as the process associated to the formal generator

$$L_{\alpha\beta} = \alpha L + \beta L^\dagger \tag{4.5}$$

where $L = L_S = \sum_{x\in\mathbb{Z}}(a_x - I)$ (3.4).

This process is constructed as follows: we *define* the semigroup acting on local functions via the series expansion of Proposition 1. This gives the finite dimensional distributions, and hence defines a unique Markov process starting from "decent" configurations, where decent means here that *both* η and $\theta\eta$ are decent in the sense of Definition 1. We call the thus defined process the "SA-process."

We then have the following.

Proposition 3. *The SA-process is monotone. As a consequence, it can be defined starting from any initial configuration.*

Proof: In Ref. 9 we constructed a generator for a coupling of the sandpile process which preserves the order. The idea of this coupling is that if $\eta \leq \xi$, then, for each site j having height two in ξ which by an addition at some site $i \in \mathbb{Z}$ could be turned into a one, in η either the height of j is one or there exists a unique site $x(j, \eta, \xi)$ having height two in η such that addition at that site creates in η a site of height one.

Let us call L_S^c the (formal) generator of this coupling. Remark now that if $\eta \leq \xi$ then of course $\theta\eta \geq \theta\xi$, and for all f monotone, Θf is also monotone.

Therefore, the coupling with generator

$$(L^c f)(\eta, \xi) = \alpha\big(L_S^c f\big)(\eta, \xi) + \beta\big(L_S^c(\Theta f)\big)(\theta\eta, \theta\xi) \tag{4.6}$$

defines a coupling that preserves the order.

This proves monotonicity. The consequence follows since every configuration can be written as an increasing (or decreasing) limit of decent configurations. $\square$

4.3. Stationary Measures for the SA Process

We denote by $\mathcal{I}$ the set of invariant measures for the SA process, of generator (4.5). We then have the following analogue of Theorem 1.

Theorem 4. *We have*

- *For* $\alpha < \beta$

$$\mathcal{I} = \{\delta_1\}$$

- *For* $\alpha > \beta$,

$$\mathcal{I} = \{\delta_2\}$$

- *For* $\alpha = \beta$

$$\mathcal{I}_e \supset \{\delta_1, \delta_2\}$$

Proof: We compute as before, starting from a translation invariant measure μ concentrating on decent configurations:

$$\int L_S \chi(\eta(0) = 1)\, d\mu = -1$$

so that, since $\chi(\eta(0) = 1) + \chi(\eta(0) = 2) = 1$,

$$\int L^{\dagger} \chi(\eta(0) = 1)\, d\mu = +1$$

Hence, starting from an initial measure μ on Ω which is translation invariant, mixing and which concentrates on decent configurations, we obtain, using (as in the proof of Theorem 2) the notation $\rho_t = (\mu S(t))(\eta(0) = 1)$:

$$\frac{d\rho_t}{dt} = -\int (\alpha L + \beta L^{\dagger})(\chi(\eta(0) = 1)) d(\mu S(t)) = (\beta - \alpha)$$

Of course this equation is only valid as long as $0 \le (\beta - \alpha)t < 1$. It expresses that the density of ones simply decreases or increases linearly until no ones are present, resp. all sites are of height one.

Notice that we used here the analogue of Proposition 1, which in this case implies that one can use the generator as in the Feller case as long as it is acting on local functions and integrated over measures with a non-zero density of sites having height two, and of sites having height one.

Starting from this equation, one concludes that for $\alpha > \beta$ (and similarly for $\alpha < \beta$), there can be no other invariant measure (which is also translation invariant) than δ_2 (resp. δ_1). We then deduce the first two statements of the theorem along the same lines as in Theorem 1, using monotonicity (by Proposition 3).

For the last statement, use that if $\alpha = \beta$,

$$\frac{d\rho_t}{dt} = 0 \tag{4.7}$$

Therefore, using standard arguments based on monotonicity, we see that δ_1 and δ_2 are invariant measures. □

Remark 5. *For the case $\alpha = \beta$ we have (4.7), i.e., the density is a conserved quantity. An open question here is whether in that case for each density there exists a stationary (in time) and ergodic (under translations) measure with that density or whether the only extremal invariant measures are $\{\delta_1, \delta_2\}$.*

In that case, we can however say the following:

1. *If μ is translation invariant, invariant for the dynamics, and with density $0 < \rho < 1$, then $\mu\Theta$ is also invariant for the dynamics. Indeed, for any decent function f, Θf is also decent, and $\int (L_S + \Theta L_S \Theta) f d\mu = 0$ is equivalent to $\int (L_S + \Theta L_S \Theta)(\Theta f) d(\mu\Theta) = 0$.*

2. *The product measure λ_ρ with density $\lambda_\rho(\eta(0) = 1) = \rho$ is not invariant. Indeed, we can proceed as in the proof of Theorem 1, and compute, for $H_n(\eta) = \chi(\eta(1) = \cdots = \eta(n) = 1)$ with $n \geq 2$,*

$$\int (L_S + L^\dagger)(H_n)\, d\lambda_\rho = \rho^{n-1}(1-\rho)(n-2) \tag{4.8}$$

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